UNIT — II ND

LAPLACE TRANSFORM :-

let $f(t)$ is fuction of all $t \geq 0$, then laplace transform of $f(t)$ is defined by laplace of $f(t) = F(S)$

$$L[f(t)] = F(S) = \int_0^\infty e^{-st} f(t)\,dt.$$

→ Some elementary Results :-

① → $L[1] = \frac{1}{S}$

② $L[t^n] = \frac{n!}{S^{n+1}} = \frac{\Gamma n+1}{S^{n+1}}$

③ → $L[e^{at}] = \frac{1}{S-a} =$

④ → $L[\sin at] = \frac{a}{S^2+a^2}$

⑤ $L[\cos at] = \frac{S}{S^2+a^2}$

⑥ $L[\sinh at] = \frac{a}{S^2-a^2}$

⑦ $L[\cosh at] = \frac{S}{S^2-a^2}$

⇒ First shifting theorem :-

OR

First Translation property :-

If $L[f(t)] = F(S)$

then $L[e^{at} f(t)] = F(S-a)$

⇒ Second translation property :-

OR

Heviside shifting theorem :-

If $L[f(t)] = FS$

and $G(t) = \begin{cases} f(t-a) ; & t>a \\ 0 ; & t<a \end{cases}$

the $L[G(t)] = e^{-as} F(S+a)$

Prob:- Find $L[f(t)]$

① if $f(t) = \begin{cases} \sin\left(t - \pi/3\right) & ; t > \pi/3 \\ 0 & ; t < \pi/3 \end{cases}$

② $f(t) = \begin{cases} (t-1)^2 & ; t > 1 \\ 0 & t < 1 \end{cases}$

Solⁿ $L[f(t)] = e^{-s} L[t^2] = e^{-s} \frac{2}{s^3}$ Ans

① $L[f(t)] = e^{-\pi/3 s} L[\sin t] = e^{-\pi/3 s} \frac{1}{s^2+1}$ Ans

Que - Find $L[e^{-t} \sin 3t]$

$L[\sin 3t] = \frac{3}{s+9}$

$L[e^{-t} \sin 3t] = \frac{3}{(s+1)^2+9} = \frac{3}{(s+1)^2+9}$ Ans

Que - 02 → $L[e^{-3t}(\sin 4t + 3\cos 4t)]$

$L[(\sin 4t + 3\cos 4t)] = \frac{4}{s^2+16} + 3\frac{s}{s^2+16}$

~~L[(sin 4t + 4~~

$L[e^{-3t}(\sin 4t + 3\cos 4t)] = \frac{4}{(s+3)^2+16} + \frac{3(s+3)}{(s+3)^2+16}$

Que - $L[e^{-3t} t^3]$

$L[t^3] = \frac{6}{s^4}$

$L[e^{-3t} t^3] = \frac{6}{(s+3)^4}$

④ $L[\cosh at \times \sin bt]$

$L\left[\frac{e^{at}+e^{-at}}{2} \sin bt\right]$

$\frac{1}{2}L[\cancel{\sin e^{at} + \sin}$ $\frac{1}{2}L\left[e^{at}\sin bt + e^{-at}\sin bt\right]$

$L[\sin bt] = \frac{b}{s^2+b^2}$

$L[e^{at}\sin bt] = \frac{b}{(s-a)^2+b^2}$

$L[e^{-at}\sin bt] = \frac{b}{(s+a)^2+b^2}$

$\frac{1}{2}L\left[e^{at}\sin bt + e^{-at}\sin bt\right] = \frac{1}{2}\left[\frac{b}{(s-a)^2+b^2} + \frac{b}{(s+a)^2+b^2}\right]$

⑤ $L[\sinh^2 2t]$

$L\left[\frac{e^{2t}-e^{-2t}}{2}\right]^2$

$\frac{1}{4}L\left[e^{4t}+e^{-4t}-2\right]$

$\frac{1}{4}\left[\frac{1}{s-4} + \frac{1}{s+4} - \frac{2}{s}\right]$ Ans

⑥ $L[1+te^{-t}]^3$

$L[1+t^3e^{-3t}+3te^{-t}+3t^2e^{-2t}]$

$L[1] + L[e^{-3t}t^3] + 3[e^{-t}t] + 3[e^{-2t}t^2]$

$\boxed{\begin{array}{l|l} L[e^{-3t}+t^3] = & 3L[e^{-2t}t^2] \\ L[t^3] = \cancel{\frac{6}{s^3}}\ \frac{6}{s^4} & 3\times\frac{2}{(s+2)^3} \\ L[e^{-3t}+t^3] = \frac{6}{(s+3)^4} & 3[e^{-t}t] = 3\frac{1}{(s+1)^2} \end{array}}$

$= \frac{1}{s} + \frac{6}{(s+3)^4} + \frac{3}{(s+1)^2} + \frac{6}{(s+3)^3}$ Ans.

⑦ $L\left[\sqrt{t}+\frac{1}{\sqrt{t}}\right]^3$ $\qquad \frac{\overline{|n+1}}{s^{n+1}}$

$= t^{3/2} + \frac{1}{t^{3/2}} + 3t\frac{1}{\sqrt{t}} + 3\sqrt{t}\,\frac{1}{t}$

$= t^{3/2} + t^{-3/2} + 3t^{1/2} + 3t^{-1/2}$

$L\left[\sqrt{t}+\frac{1}{\sqrt{3}}\right]^3 = L\left[t^{3/2}+t^{-3/2}+3t^{1/2}+3t^{-1/2}\right]$

$= L[t^{3/2}] + L[t^{-3/2}] + 3L[t^{1/2}] + 3L[t^{-1/2}]$

$= \frac{\overline{|5/2}}{s^{5/2}} + \frac{\overline{|-1/2}}{s^{-1/2}} + 3\frac{\overline{|3/2}}{s^{3/2}} + 3\frac{\overline{|1/2}}{s^{1/2}}$

$= \frac{\frac{3}{2}\cdot\frac{1}{2}\overline{|\tfrac{1}{2}}}{s^{5/2}} + \frac{-2\sqrt{\pi}}{s^{-1/2}} + 3\frac{\frac{1}{2}\overline{|\tfrac{1}{2}}}{s^{3/2}} + \frac{3\sqrt{\pi}}{s^{1/2}}$

$\frac{3\sqrt{\pi}}{4s^{5/2}} + \frac{-2\sqrt{\pi}}{s^{-1/2}} + \frac{3\sqrt{\pi}}{2s^{3/2}} + \frac{3\sqrt{\pi}}{s^{1/2}}$

Laplace transform of Integrals

if $L[f(t)] = F(S)$ then $L\left[\int_0^t f(t)dt\right] = \frac{1}{S}F(S)$

Find → $L\left[\int_0^t e^{at}\sin bt\, dt\right]$

$$L[\sin bt] = \frac{b}{s^2+b^2}$$

$$L[e^{at}\sin bt] = \frac{b}{(s-a)^2+b^2}$$

$$L\left[\int_0^t e^{at}\sin bt\, dt\right] = \frac{1}{S}\frac{b}{(S-a)^2+b^2}$$

* Multiplication by t^n

$L[f(t)] = Fs$ then $L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}F(s)$

* Division by t :-

$L[f(t)] = F(s)$ then $L\left[\frac{f(t)}{t}\right] = \int_S^\infty F(s)ds$

* Find laplace transform *

Que - 01 → $\int_0^t \sin 3t \cosh 2t\, dt$.

$$L\left[\int_0^t \sin 3t \cosh 2t\, dt\right]$$

$$L[\sin 3t] = \frac{3}{s^2+9}$$

$$L[\cosh 2t] = L\left[\frac{e^{2t} - e^{-2t}}{2} \sin 3t\right] = \frac{1}{2}\left[e^{2t}\sin 3t - e^{-2t}\sin 3t\right]$$

$$= \frac{1}{2}\left[\frac{3}{(S-2)^2+9} - \frac{3}{(S+2)^2+9}\right] = f(S)$$

$$L\left[\int_0^t \sin 3t \cosh 2t\, dt\right] = \frac{3}{2s}\left[\frac{1}{(s-2)^2+9} - \frac{1}{(s+2)^2+9}\right]$$

(ii) $te^{-t}\sin 2t$

$L[te^{-t}\sin 2t]$

$$L[\sin 2t] = \frac{2}{s^2+4}$$

$$\cancel{L[t]} \quad \frac{1}{s^2} \qquad L[t\sin t] \; \frac{(-1)^1 d}{ds} \frac{2}{s^2+4}$$

$$= -2 \frac{-1}{(s^2+4)^2} 2s$$

$$= \frac{4s}{(s^2+4)^2}$$

$$L[e^{-t} t \sin t] = \frac{4(s+1)}{[(s+1)^2+4]^2} \quad \text{Ans}$$

(iii) $t^2 e^t \sin 4t$

$L[t^2 e^t \sin 4t]$

$$L[\sin 4t] = \frac{4}{s^2+16}$$

$$L[t^2 \sin 4t] = \cancel{-1}^2 \frac{d^2}{ds^2} \frac{4}{s^2+16}$$

$$= 4\frac{d}{ds} \frac{-1}{(s^2+16)^2} 2s$$

$$= \cancel{4} -8 \frac{d}{ds} \frac{s}{(s^2+16)^2}$$

$$= -8 \frac{(s^2+16)^2 - s\,2(s^2+16)\,2s}{(s^2+16)^{4}}$$

$$= -8 \frac{(s^2+16) - 4s^2}{(s^2+16)^3}$$

$$= \cancel{-8}\; -8 \frac{-3s^2+16}{(s^2+16)^3}$$

$$L[t^2 e^t \sin 4t] = -8 \frac{-3(s-1)^2+16}{[(s-1)^2+16]^3} \quad \text{Ans}$$

$$\frac{u}{v} = \frac{vu' - v'u}{v^2}$$

(iv) $\dfrac{1-\cos t}{t}$

$L\left[\dfrac{1-\cos t}{t}\right]$

$L[1-\cos t] = \dfrac{1}{s} - \dfrac{s}{s^2+1}$

$L\left[\dfrac{1-\cos t}{t}\right] = \int_s^\infty \left[\dfrac{1}{s} - \dfrac{1}{2}\dfrac{2s}{s^2+1}\right] ds$

$= \left[\log s - \dfrac{1}{2}\log(s^2+1)\right]_s^\infty$

$= \left[\log s - \log\sqrt{s^2+1}\right]_0^\infty$

$= \left[\log \dfrac{s}{\sqrt{s^2+1}}\right]_0^\infty$

$= -\log \dfrac{s}{\sqrt{s^2+1}}$

Ans

(5) Find laplace transform of following

(1) $\dfrac{e^{at} - \cos bt}{t}$ (2) $\dfrac{1-\cos t}{t^2}$ (3) $\dfrac{\sin at}{t}$ (4) $\dfrac{e^{-at} - e^{-bt}}{t}$

(5) $\dfrac{1-\cos 2t}{t}$ (6) $\dfrac{\cos at - \cos bt}{t}$ (7) $\dfrac{e^{-t}\sin t}{t}$ (8) $\int_0^{t/2} \dfrac{1-e^{-2x}}{x}$

(9) $\int_0^t e^{-t}\dfrac{\sin^2 t}{t}\,dt$ (10) $\int_0^t \dfrac{1}{u} e^{-4u} \sin 3u\,du$ (11) $e^{-4t}\int_0^t \dfrac{\sin 3t}{t}\,dt$

(12) $\int_0^t\int_0^t\int_0^t \cos at\,dt\,dt\,dt.$

Que - 6 $\quad L\left[\frac{\cos at - \cos bt}{t}\right]$

$$L[\cos at - \sin bt] = \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}$$

$$L\left[\frac{\cos at - \cos bt}{t}\right] = \int_s^\infty \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}$$

$$= \frac{1}{2}\int_s^\infty \left[\frac{2s}{s^2+a^2} - \frac{2s}{s^2+b^2}\right]$$

$$= \frac{1}{2}\left[\log(s^2+a^2) - \log(s^2+b^2)\right]_s^\infty$$

$$\frac{1}{2}\left[\log \frac{s^2+a^2}{s^2+b^2}\right]_s^\infty$$

$$-\frac{1}{2}\log\frac{s^2+a^2}{s^2+b^2}$$

$$\log\sqrt{\frac{s^2+b^2}{s^2+a^2}}$$

Que - 01 :- $\quad L[e^{at} - \cos bt]$

$$L[e^{at}] = \frac{1}{s-a}$$

$$L[\cos bt] = \frac{s}{s^2+b^2}$$

$$L[e^{at} - \cos bt] = \int_s^\infty \left(\frac{1}{s-a} - \frac{s}{s^2+b^2}\right) ds$$

$$= \int_s^\infty \left(\frac{1}{s-a} - \frac{1}{2}\frac{2s}{(s^2+b^2)}\right) ds = \left[\log(s-a) - \frac{1}{2}\log(s^2+b^2)\right]_s^\infty$$

$$= \left[\log\frac{(s-a)}{\sqrt{(s^2+b^2)}}\right]_s^\infty$$

$$-\log \frac{s-a}{\sqrt{s^2+b^2}}$$

$$= \log \frac{\sqrt{s^2+a^2}}{s-a} \quad \text{Ans}$$

② $L\left[\frac{1-\cos 2t}{t}\right]$

$$L[1-\cos 2t] = \frac{1}{s} - \frac{s}{s^2+4}$$

$$L\left[\frac{1-\cos 2t}{t}\right] = \int_s^\infty \left(\frac{1}{s} - \frac{1}{2}\frac{2s}{s^2+4}\right) ds$$

$$\int_s^\infty \left(\frac{1}{s} - \frac{1}{2}\frac{2s}{s^2+4}\right) ds = \left[\log s - \frac{1}{2}\log s^2+4\right]_s^\infty$$

$$\left[\log \frac{s}{\sqrt{s^2+4}}\right]_s^\infty = \log \frac{\sqrt{s^2+4}}{s} \quad \text{Ans}$$

⑤ $L\left[\frac{e^{-at} - e^{-bt}}{t}\right]$

$$L[e^{-at} - e^{-bt}] = \frac{1}{s+a} - \frac{1}{s+b}$$

$$L\left[\frac{e^{-at} - e^{-bt}}{t}\right] = \int_s^\infty \left(\frac{1}{s+a} - \frac{1}{s+b}\right) ds$$

$$= [\log s+a - \log s+b]_s^\infty$$

$$= \left[\log \frac{s+a}{s+b}\right]_s^\infty = -\log \frac{s+a}{s+b}$$

$$= \log \frac{s+b}{s+a} \quad \text{Ans}$$

⑧ $L\left[\cancel{\frac{\cos at - \cos bt}{t}}\right]$

$\cancel{\cos at - \cos b}$

⑨ $\int_0^t e^{-t} \frac{\sin^2 t}{t} \, d$

$\sin^2 t = \frac{1-\cos 2t}{2}$

$L\left[\frac{1-\cos 2t}{2t}\right] = \log \frac{\sqrt{s^2+4}}{s}$

$L\left[\cancel{e^{-t}\frac{\sin^2 t}{t}}\right] = \cancel{\log \frac{\sqrt{(s+1)^2+4}}{s+1}}$

$\frac{1}{2} L[1-\cos 2t] = \frac{1}{2} \log \frac{\sqrt{s^2+4}}{s}$

$\frac{1}{2} L\left[e^{-t} \frac{\sin^2 t}{t}\right] = \frac{1}{2}\int_0^t \log \frac{\sqrt{(s+1)^2+4}}{(s+1)}$

$= \cancel{\int_0^t \log \frac{[(s+1)^2+4]^{-3/2}}{\sqrt{(s+1)}}}$

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① → $t\cosh t$

② → $t(3\sin 2t - 2\cos 2t)$

③ → $t\sin^2 3t$

④ → $t^2 e^{t} \sin 4t$

⑤ → $t^3 e^{-2t}$

⑥ → $f(x) = x^3 \sin x$

⑤ $L[t^3 e^{-2t}]$

$L[t^3] = \frac{3!}{s^4} = \frac{6}{s^4}$

$L[t^3 e^{-2t}] = \frac{6}{(s+2)^4}$

① $L[t \cdot \cosh 3t]$

$$\frac{u}{v} = \frac{vu' - uv'}{v^2}$$

$$L\left[t\left(\frac{e^{3t} + e^{-3t}}{2}\right)\right]$$

$$L\left[t\left(\frac{e^{3t}}{2} + \frac{e^{-3t}}{2}\right)\right]$$

$$L[t] = \frac{1}{s^2}$$

$$L\left[t\left(\frac{e^{3t}}{2} + \frac{e^{-3t}}{2}\right)\right] = \frac{1}{2}\left[\frac{1}{(s-3)^2} + \frac{1}{(s+3)^2}\right]$$ Ans

② $L[t(3\sin 2t - 2\cos 2t)]$

$L[~~t(3\sin 2t - 2\cos 2t)~~]$

$L[(3t\sin 2t - 2t\cos 2t)]$

$$L[\sin 2t] = \frac{2}{s^2+4} = \frac{2}{s^2+4} = Fs$$

$$L[3t\sin 2t] = 3\left[\frac{-1\, d}{ds} \cdot \frac{2}{s^2+4}\right]$$

$$= -3 \times -\frac{2}{(s^2+4)^2} 2s$$

$$L[3t\sin 2t] = \frac{12s}{(s^2+4)^2}$$

$$L[\cos 2t] = \frac{s}{s^2+4}$$

$$L[2t\cos 2t] = (-1)\frac{d}{ds}\frac{s}{s^2+4}$$

$$L[2t\cos 2t] = 2\left[\frac{(s^2+4)1 - s(2s)}{(s^2+4)^2}\right] = 2\left[\frac{(s^2+4) - 2s^2}{(s^2+4)^2}\right]$$

$\cancel{2s^2+8}$ -

$L[t(\sin 2t - 2\cos 2t)]$

$$= \frac{12s}{s^2+4} - 2\left[\frac{(s^2+4) - s^2}{(s^2+4)^2}\right]$$

$$= \frac{12s}{s^2+4} - \frac{2s^2 - 8 + s^2}{(s^2+4)^2} \quad \text{Ans}$$

(4) $t^2 e^t \sin 4t$

⑥ $L[\sin x] = \dfrac{1}{s^2+1}$

$L[x^3 \sin x] = (-1)^3 \dfrac{d^3}{ds^3} \dfrac{1}{s^2+1}$

$= -\dfrac{d^2}{ds^2} \dfrac{-2s}{(s^2+1)^2}$

$= 2\dfrac{d}{ds} \dfrac{(s^2+1)^2 \times 1 - s\,2(s^2+1)\,2s}{(s^2+1)^4}$ $\cancel{L[x^3 \sin x]}$

$= \dfrac{2d}{ds} \dfrac{s^2+1-4s^2}{(s^2+1)^3}$

$= 2\dfrac{d}{ds} \dfrac{1-3s^2}{(s^2+1)^3}$

$= 2\,\dfrac{(s^2+1)^3(-6s) - (1-3s^2)\;3(s^3+1)^2\,2s}{(s^2+1)^6}$

$= -2\cancel{(s^2+1)^2}\,6s \;\dfrac{s^2+1+1-3s^2}{(s^2+1)^{\cancel{6}\,4}}$

$= -12s \times \dfrac{2-2s^2}{(s^2+1)^4}$

$= \dfrac{24s(s^2-1)}{(s^2+1)^4}$ Ans

Evaluation of Integrals

$$L[f(t)] = F(s) = \int_0^\infty e^{-st} f(t)\, dt.$$

Que-01 $\int_0^\infty e^{-3t} \frac{\sin t}{t} dt$ Evaluate this

Sol^n $L[\sin t] = \frac{1}{s^2+1^2} = \frac{1}{s^2+1}$

$$L\left[\frac{\sin t}{t}\right] = \int_s^\infty \frac{1}{s^2+1} = \left[\tan^{-1} s\right]_s^\infty$$

$$= \tan^{-1}\infty - \tan^{-1} s$$

$$\frac{\pi}{2} - \tan^{-1} s$$

$$\cot^{-1} s$$

$$\int_0^\infty e^{-3t} \frac{\sin t}{t} dt = \cot^{-1}(3)$$

Que-03 :- Evaluate $\int_0^\infty \frac{\cos 6t - \cos 4t}{t} dt.$

Sol $-$ $L[\cos 6t] = \frac{s}{s^2+36}$

$$L[\cos 4t] = \frac{s}{s^2+16}$$

$$L\left[\frac{\cos 6t - \cos t}{t}\right] ds = \int_s^\infty \left(\frac{s}{s^2+36} - \frac{s}{s^2+16}\right) ds$$

$$= \frac{1}{2} \int_s^\infty \left(\frac{2s}{s^2+36} - \frac{2s}{s^2+16}\right) ds$$

$$= \frac{1}{2}\left[\log(s^2+36) - \log(s^2+16)\right]_s^\infty$$

$$= \frac{1}{2}\left(\log \frac{s^2+36}{s^2+16}\right)_s^\infty = -\frac{1}{2} \log \frac{s^2+36}{s^2+16}$$

$$-\log\sqrt{\frac{s^2+6^2}{s^2+4^2}} = \log\sqrt{\frac{s^2+4^2}{s^2+6^2}}$$

$$\int_0^\infty e^{-0t}\,\frac{\cos 6t - \cos 4t}{t}\,dt = \log\sqrt{\frac{16}{36}} = \log\frac{4}{6}$$

$$= \log\frac{2}{3} \quad \underline{Ans}$$

Que-06 Evaluate $\int_0^\infty \int_0^t e^{-t}\,\frac{\sin u}{u}\,du\,dt = \frac{\pi}{4}$

$$L[\sin u] = \frac{1}{s^2+1}$$

$$L\left[\frac{\sin u}{u}\right] = \int_s^\infty \frac{1}{s^2+1}\,ds = \left[\tan^{-1} s\right]_s^\infty$$

$$= \tan^{-1}\infty - \tan^{-1}s$$

$$= \frac{\pi}{2} - \tan^{-1}s$$

$$= \cot^{-1}s$$

$$\int_0^\infty e^{-t}\left[\int_0^t \frac{\sin u}{u}\,du\right]dt = L\left[\int_0^t \frac{\sin u}{u}\,du\right] = f = \frac{1}{s}\cot^{-1}s$$

$$L\left[\int_0^t \frac{\sin u}{u}\,du\right] = \frac{1}{s}\cot^{-1}s$$

$$\int_0^\infty e^{-t}\left(\int_0^t \frac{\sin u}{u}\,du\right)dt = \frac{1}{1}\cot^{-1}(1)$$

$$= \frac{\pi}{4}$$

Que-02 Evaluate $\int_0^\infty \frac{e^{-at} - e^{-bt}}{t}\,dt$

$$L[e^{-at} - e^{-bt}] = \frac{1}{s+a} - \frac{1}{s+b}$$

$$L\left[\frac{e^{-at} - e^{-bt}}{t}\right] = \int_s^\infty \left(\frac{1}{s+a} - \frac{1}{s+b}\right)ds$$

$$\left[\log(s+a) - \log(s+b)\right]_s^\infty = \left[\log\frac{s+a}{s+b}\right]_s^\infty$$

$$= -\log\frac{s+a}{s+b} = \log\frac{s+b}{s+a}$$

$$\int_0^\infty e^{-0t}\,\frac{e^{-at}-e^{-bt}}{t} = \log\frac{s+b}{s+a} = \underline{\log\frac{b}{a}}\text{ Ans}$$

Que-04 Evaluate $\int_0^\infty e^{-2t}\,t^2 \sin 3t\,dt$.

$$L[\sin 3t] = \frac{3}{s^2+9}$$

$$L[t^2 \sin 3t] = 3\left[(-1)^2 \frac{d^2}{ds^2}\frac{3}{s^2+9}\right]$$

$$= -3\,\frac{d}{ds}\,\frac{2s}{(s^2+9)^2}$$

$$= -6\left[\frac{(s^2+9)^2\times 1 - 2s(2s)}{(s^2+9)^4}\right]$$

$$= -6\left[\frac{s^2+9^2 - 2s^2}{s^2}\right.$$

Ans $\frac{18}{2197}$

Laplace Transform of Periodic function :-

$$f(t+T) = f(t)$$

$$\sin(\theta+2\pi) = \sin\theta \quad \text{periodic}$$

If $f(t)$ periodic function of time i.e.

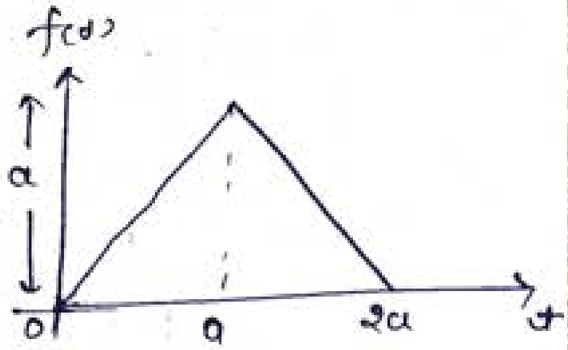
$$f(t+T) = f(t)$$

then laplace transform $L\left[f(t)\right] = \dfrac{1}{1-e^{-sT}}\displaystyle\int_0^T e^{-st} f(t)\, dt$

Que-02 Draw the graph and find the laplace transform of the triangular wave function of period 2π given by

$$f(t) \begin{cases} t & , 0 < t \le \pi \\ 2\pi - t & \pi < t < 2\pi \end{cases}$$

Soln :- laplace transform of periodic function



$$L\left[f(t)\right] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\, dt$$

$$L\left[f(t)\right] = \frac{1}{1-e^{-2\pi s}}\int_0^{2\pi} e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{-2\pi s}}\left[\int_0^{\pi} t e^{-st}\, dt + \int_{\pi}^{2\pi} (2\pi - t) e^{-st}\, dt\right]$$

$$= \frac{1}{1-e^{-2\pi s}}\left[\left\{\frac{te^{-st}}{-s} - \frac{e^{-st}}{s^2}\right\}_0^{\pi} + \left\{(2\pi - t)\frac{e^{-st}}{-s} - (-1)\frac{e^{-st}}{s^2}\right\}_{\pi}^{2\pi}\right]$$

$$= \frac{1}{1-e^{-2\pi s}}\left[\cancel{-\frac{\pi e^{-\pi s}}{s}} - \frac{e^{-\pi s}}{s^2} + \frac{1}{s^2} + \frac{e^{-2\pi s}}{s^2} + \cancel{\frac{\pi e^{-\pi s}}{s}} - \frac{e^{-\pi s}}{s^2}\right]$$

$$= \frac{1}{s^2(1-e^{-2\pi s})}\left[-2e^{-\pi s} + 1 + e^{-2\pi s}\right]$$

$$= \frac{1}{\cancel{s^2(1-e^{-\pi s})}(1+e^{-\pi s})} \cancel{(1-e^{-\pi s})^2}$$

$$L[f(t)] = \frac{1-e^{-\pi s}}{s^2(1+e^{-\pi s})}$$

Que-05 :- Determine the laplace transform of the periodic function defined by the triangular wave function of period $2a$ $f(t)$

$$= \begin{cases} \frac{t}{a} & \text{for } 0 \le t \le a \\ \frac{2a-t}{a} & \text{for } a \le t \le 2a \end{cases}$$

Soln =

laplace transform of periodic function.

$$L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt$$

$T = 2a$

$$= \frac{1}{1-e^{-2as}} \int_0^{2a} e^{-2at} f(t)\,dt$$

$$= \frac{1}{(1-e^{-2as})} \left[\int_0^a \frac{t e^{-st}}{a}\,dt + \int_a^{2a} \left(\frac{2a-t}{a}\right) e^{-st}\,dt \right]$$

$$= \frac{1}{a(1-e^{-2as})} \left[\left\{ \frac{te^{-st}}{-s} - \frac{e^{-st}}{s^2} \right\}_0^a + \left\{ \frac{(2a-t)e^{-st}}{-s} - (-1)\frac{e^{-st}}{s^2} \right\}_a^{2a} \right]$$

$$= \frac{1}{a(1-e^{-2as})} \left[\cancel{-\frac{ae^{-as}}{-s}} - \frac{e^{-as}}{s^2} + \frac{1}{s^2} + \frac{e^{-2as}}{s^2} + \cancel{\frac{ae^{-as}}{s}} - \frac{e^{-as}}{s^2} \right]$$

$$= \frac{1}{s^2 a(1-e^{-2as})} \left[2ae^{-as} + 1 + e^{-2as} \right]$$

$$= \frac{1}{\cancel{as^2(1-e^{-as})}(1+e^{-as})} \cancel{(1-e^{-as})^2}$$

$$L[f(t)] = \frac{1-e^{-as}}{as^2(1+e^{-as})}$$

Que - 03 :- Find the laplace transform of square wave function of period a defined as.

$$f(t) = \begin{cases} 1, & 0 \le t \le a/2 \\ -1 & a/2 \le t < a \end{cases}$$

Sol^n $L[f(t)] = \cancel{\frac{1}{1-e^{-sa}}} \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt.$

$T = a$

$$L[f(t)] = \frac{1}{1-e^{-as}} \int_0^a e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{-as}} \left[\int_0^{a/2} e^{-st} dt + \int_{a/2}^a -e^{-st}\, dt \right]$$

$$= \frac{1}{1-e^{-as}} \left[\left\{ \frac{e^{-st}}{-s} \right\}_0^{a/2} + \left\{ \frac{e^{-st}}{s} \right\}_{a/2}^a \right]$$

$$= \frac{1}{(1-e^{-as})} \left[\frac{e^{-a/2\, s}}{-s} + \frac{1}{s} + \frac{e^{-as}}{s} - \frac{e^{-a/2\, s}}{s} \right]$$

$$= \frac{1}{s(1-e^{-as})} \left[-2e^{-a/2\, s} + 1 + e^{-as} \right)$$

$$= \frac{1}{s(1-e^{-as/2})(1+e^{-as/2})} \times (1-e^{-as/2})^2$$

$$= \frac{1-e^{-as/2}}{s(1+e^{-as/2})} \quad \text{Ans}$$

$$L[f(t)] = \frac{1-e^{-as/2}}{s(1+e^{-as/2})} \quad \text{Ans}$$

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin(bx) - b\cos(bx)\right] + C$$

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos(bx) + b\sin(bx)\right] + C$$

(4) A periodic function is defined by $f(t) = \begin{cases} \sin\omega t & 0 \le t \le \pi/\omega \\ 0 & \pi/\omega \le t \le 2\pi/\omega \end{cases}$

$\cancel{L[f(t)] = \frac{1}{1-e^{-sT}}}$

$$L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt$$

$$= \cancel{\frac{1}{1-e^{-s2\pi/\omega}}}$$

$$\frac{1}{1-e^{-s2\pi/\omega}} \int_0^{2\pi/\omega} \left[e^{-st} f(t)\,dt\right]$$

$$\frac{1}{1-e^{-2\pi s/\omega}} \left[\int_0^{\pi/\omega} \sin\omega t\, e^{-st}\,dt + \int_{\pi/\omega}^{2\pi/\omega} e^{-st}\cdot 0\,dt\right]$$

$$\cancel{\frac{1}{1-e^{-2\pi/\omega}}}\ \frac{1}{1-e^{-2\pi s/\omega}} \left[\frac{e^{-st}}{s^2+\omega^2}\left\{-s\sin\omega t - \omega\cos\omega t\right\}\Big|_0^{\pi/\omega}\right]$$

$$\frac{1}{\left(1-e^{-\frac{2\pi s}{\omega}}\right)} \left[\frac{-e^{-st}}{s^2+\omega^2}\left\{s\sin\omega t + \omega\cos\omega t\right\}\right]_0^{\pi/\omega}$$

$$\cancel{\frac{1}{1-e^{-2s\pi/\omega}}\left[\frac{-e^{-st}}{}\right.}$$

$$\frac{1}{\left(1-e^{-\frac{2\pi s}{\omega}}\right)(s^2+\omega^2)} \left\{-e^{-\pi s/\omega}\, s\sin\omega\frac{\pi}{\omega} - e^{-s\pi\omega}\,\omega\cos\omega\frac{\pi}{\omega}\right\}$$

$$\left(\frac{1}{1-e^{-\frac{2\pi s}{\omega}}}\right)(s^2+\omega^2)\left[\omega t + \omega e^{-s\pi/\omega}\right]$$

Ans

Inverse Laplace Transform :-

$\Rightarrow \quad L[f(t)] = f(s)$

$L^{-1}[f(s)] = f(t)$

① $L^{-1}\left[\frac{1}{s}\right] = 1$

② $L^{-1}\left[\frac{1}{s^n}\right] = \frac{t^{n-1}}{(n-1)!}$

③ $L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$

④ $L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cancel{\cos} \cos at$

⑤ $L^{-1}\left[\frac{s}{s^2-a^2}\right] = \cosh at$

⑥ $L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at$

⑦ $L^{-1}\left[\frac{1}{s^2-a^2}\right] = \frac{1}{a}\sinh at$

First Shifting Property :-

If laplace inverse $f(s)$

$L^{-1}[f(s)] = f(t)$

$L^{-1}[f(s-a)] = e^{at} f(t)$

Que-1 Find the laplace inverse transform of

① $\frac{s+8}{s^2+4s+5}$

② $\frac{1}{s^2+2s+2}$

③ $\frac{1}{s^2-3s+3}$

① $\frac{s+8}{s^2+4s+5} = \frac{s+8}{s^2+2\times2\times s+5-2^2+2^2} = \frac{s+2+6}{(s+2)^2+9}$

$L^{-1}\left[\frac{(s+2+6}{(s+2)^2+1^2}\right] = \left[\frac{s+2}{(s+2)^2+1^2} + \frac{6}{(s+2)^2+1^2}\right] = e^{-2t}\cos t + 6e^{-2t}\sin t$

$L^{-1}\left[\frac{s+8}{s^2+4s+5)}\right] = e^{-2t}\left[\cos t + 6\sin t\right]$

ii) $L^{-1}\left[\frac{1}{s^2+2s+2}\right]$

making Perfect square in the

$L^{-1}\left[\frac{1}{(s+1)^2+1^2}\right] = e^{-t}\sin t$

iii) $L^{-1}\left[\frac{1}{s^2-3s+5}\right]$

making perfect square in the

$L^{-1}\left[\frac{1}{(s-\frac{3}{2})^2+(\frac{\sqrt{3}}{2})^2}\right] = e^{3/2t}\,\frac{2}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t.$

* laplace transform of Derivative :-
Inverse

If Laplace inverse $f(s) = f(t)$ then laplace inverse $\left[\frac{d^n}{ds^n} F(s)\right]$

$$= (-1)^n t^n f(t)$$

OR $\boxed{L^{-1}\left[\frac{d}{ds} F(s) = -t f(t)\right]}$

note - Use in case of log, $\cot^{-1}$, $\tan^{-1}$

$\boxed{\text{Que- iii} \quad \log \frac{s+a}{s+b}}$

Que -3 find the function whose laplace transform is $\log\left(1+\frac{1}{s^2}\right)$

$$= \log\left(1+\frac{1}{s^2}\right)$$

$$= \log\left(\frac{s^2+1}{s^2}\right)$$

$$= \log s^2+1 - \log s^2$$

$$L^{-1}\left[\frac{d}{ds}\{\log(s^2+1) - \log s^2\}\right] = -t f(t)$$

$$L^{-1}\left[\frac{2s}{s^2+1} - \frac{2s}{s^2}\right] = -t f(t)$$

$$2L^{-1}\left[\frac{s}{s^2+1} - \frac{1}{s}\right] = -t f(t)$$

$$2(\cos t - 1) = -t f(t)$$

$$f(t) = \frac{2(\cos t - 1)}{-t}$$

$$f(t) = \frac{2(1-\cos t)}{t} \text{ Ans}$$

(ix) $\log\left(\frac{s+a}{s+b}\right)$

$= \log(s+a) - \log(s+b)$

$L^{-1}\left[\frac{d}{ds}\{\log s+a - \log s+b\}\right] = -t f(t)$

$L^{-1}\left[\frac{1}{s+a} - \frac{1}{s+b}\right] = -t f(t)$

$e^{-at} - e^{-bt} = -t f(t)$

$f(t) = \frac{e^{-at} - e^{-bt}}{-t}$

$f(t) = \frac{e^{-bt} - e^{-at}}{t}$

(2) Find the function whose transform is

$F(s) = \log\frac{(s^2+1)}{s(s+1)}$

Sol^n $\log\frac{(s^2+1)}{s(s+1)}$

$\log(s^2+1) - \log(s) - \log(s+1)$

$L^{-1}\left[\frac{d}{ds}\{\log s^2+1 - \log s - \log s+1\}\right] = -t f(t)$

$L^{-1}\left[\frac{2s}{s^2+1} - \frac{1}{s} - \frac{1}{s+1}\right] = -t f(t)$

$2\cos t - 1 - e^{-t} = -t f(t)$

$f(t) = \frac{2\cos t - 1 - e^{-t}}{-t}$

$f(t) = \frac{1 + e^{-t} - 2\cos t}{t}$ Ans

(vii) $\cot^{-1}\left(\frac{s+3}{2}\right)$

$$L^{-1}\left[\frac{d}{ds}\left\{\cot^{-1}\frac{s+3}{2}\right\}\right] = -t f(t)$$

$$L^{-1}\left[+\frac{1}{1+\left(\frac{s+3}{2}\right)^2}\,\frac{1}{2}\right] = +t f(t)$$

$$\frac{1}{\cancel{2}}L^{-1}\left[\frac{\cancel{4}}{4+(s+3)^2}\right] = t f(t)$$

$$L^{-1}\left[\frac{2}{4+(s+3)^2}\right] = t f(t)$$

$$e^{-3t}L^{-1}\left[\frac{2}{s^2+2^2}\right] = t f(t)$$

$$e^{-3t}\sin 2t = t f(t)$$

(viii) $\cot^{-1}\left(\frac{s}{2}\right)$

$$L^{-1}\left[\frac{d}{ds}\left\{\cot^{-1}\left(\frac{s}{2}\right)\right\}\right] = -t f(t)$$

$$L^{-1}\left[+\frac{1}{1+\left(\frac{s}{2}\right)^2}\,\frac{1}{2}\right] = \cancel{+}t f(t)$$

$$\frac{1}{\cancel{2}}L^{-1}\left[\frac{\cancel{4}}{4+s^2}\right] = t f(t)$$

$$L^{-1}\left[\frac{2}{2^2+s^2}\right] = t f(t)$$

$$\boxed{f(t) = \frac{\sin 2t}{t}}$$

Ans

3

<u>Que</u> $\frac{3}{s^2+2s-8} =$

$\Rightarrow \frac{3}{s^2+2\times1\times s+1^2-1-8} = \frac{3}{(s+1)^2-9}$

$L^{-1}\left[\frac{3}{(s+1)^2-3^2}\right]$

$e^{-t}L^{-1}\left[\frac{3}{s^2-3^2}\right]$

$\cancel{3}e^{-t}\ \frac{1}{\cancel{3}}\ \sinh 3t$

$= \underline{e^{-t}\sinh 3t}$

<u>Que</u> $\frac{1}{(s+3)^4} = L^{-1}\left[\frac{1}{(s+3)^4}\right]$

$= e^{-3t}\ L^{-1}\left[\frac{1}{s^4}\right]$

$= e^{-3t}\ \frac{t^{4-1}}{(4-1)!}$

$= e^{-3t}\ \frac{t^3}{6}$

Ans.

Que - find inverse Laplace transform following.

(i) $\log \frac{P(P+1)}{P^2+4}$ (iv) $\log\left\{1-\frac{a^2}{s^2}\right\}$

(ii) $\log\left\{\frac{s+1}{(s+2)(s+3)}\right\}$

(iii) $\log\left\{\frac{s^2+1}{s(s+1)}\right\}$

$2\sin A \cos B = \sin(A-B) + \sin(A+B)$

$2\cos A \cos B = \cancel{\cos(A+B)}$

$2\sin A \sin B =$

(i) $\log\frac{P(P+1)}{P^2+4} = L^{-1}\left[\frac{d}{ds}\left\{\log(P)+\log(P+1)-\log(P^2+4)\right\}\right] = -tf(t)$

$= L^{-1}\left[\frac{1}{P}+\frac{1}{P+1}-\frac{2P}{P^2+2^2}\right] = -tf(t)$

$= 1+\sin t - 2\cos 2t = -tf(t)$

$f(t) = -\frac{1+\sin t - 2\cos 2t}{t}$ Ans

(ii) $\log\left\{\frac{s+1}{(s+2)(s+3)}\right\}$

$L^{-1}\left[\frac{d}{ds}\left\{\log(s+1)-\log(s+2)-\log(s+3)\right\}\right] = -tf(t)$

$L^{-1}\left[\frac{1}{s+1}-\frac{1}{s+2}-\frac{1}{s+3}\right] = -tf(t)$

$\left[e^{-t}-e^{-2t}-e^{-3t}\right] = -tf(t)$

$f(t) = \frac{\left(-e^{-t}+e^{-2t}+e^{-3t}\right]}{t}\cdots$

Convolution Theorem

Statement – If laplace inverse $F(s) = f(t)$

and $L^{-1}[G(s)] = g(t)$

then $$\boxed{L^{-1}[F(s)\cdot G(s)] = \int_0^t f(u)\, g(t-u)\, du}$$

$\cos(A-B) - \cos(A+B)$

$2\sin A \cos B = \cancel{\cos(A-B)} +$

$2\cos A \sin B = \cos(A+B) + \cos(A-B)$

$2\sin A \sin B = \sin(A-B) + \sin(A+B)$

$2\cos A \cos B \quad \sin(A+B) + \sin(A-B)$

Qu-1 Use convolution theorem to evaluate $L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\}$

let $F(s) = \frac{s}{s^2+a^2}$

$G(s) = \frac{s}{s^2+b^2}$

$L^{-1}[F(s)] = \cos at = f(t)$

$L^{-1}[G(s)] = \cos bt = g(t)$

Convolution theorem

$$L^{-1}[F(s)\cdot G(s)] = \int_0^t f(u)\, g(t-u)\, du$$

$$= \frac{1}{2}\int_0^t 2\cos au \cos(bt-bu)\, du$$

$$= \frac{1}{2}\int_0^t \left[\cos(au+bt-bu) + \cos(au-bt+bu)\right] du$$

$$= \frac{1}{2}\left[\frac{\sin(au+bt-bu)}{(a-b)} + \frac{\sin(au-bt+bu)}{a+b}\right]_0^t$$

$$= \frac{1}{2}\left[\frac{\sin(at)}{a-b} + \frac{\sin at}{a+b} - \frac{\sin(bt)}{a-b} + \frac{\sin(+bt)}{a+b}\right]$$

$\sin(-\theta) = -\sin\theta$

$$\frac{1}{2}\left[\sin at\left(\frac{1}{a-b}+\frac{1}{a+b}\right)+\sin bt\left(\frac{1}{a+b}-\frac{1}{a-b}\right)\right]$$

$$\frac{1}{2}\left[\sin at\left(\frac{a+b+a-b}{a^2-b^2}\right)+\sin bt\left(\frac{a-b-a-b}{a^2-b^2}\right)\right]$$

$$\frac{1}{2(a^2-b^2)}\left[2a\sin at-2b\sin bt\right]$$

$$\left[\frac{a\sin at-b\sin bt}{a^2-b^2}\right] \text{ Ans}$$

Que-1/ Use convolution theorem to evaluate

$$L^{-1}\left\{\frac{s}{(s^2+1)(s^2+4)}\right\}$$

let $F(s)=\frac{1}{s^2+1}$

$G(s)=\frac{s}{s^2+2^2}$

$L^{-1}[F(s)]=\sin t=f(t)$

$L^{-1}[G(s)]=\cos 2t=g(t)$

$$L^{-1}[F(s)\cdot G(s)]=\int_0^t f(u)\,g(t-u)\,du$$

$$=\int_0^t[\sin u\cdot\cos(2t-2u)]\,du$$

$$=\frac{1}{2}\int_0^t[2\sin u\cdot\cos(2t-2u)]\,du$$

$$=\frac{1}{2}\int_0^t[\sin(u-2t+2u)+\sin(u+2t-2u)]\,du$$

$$\frac{1}{2}\left[-\frac{\cos(u-2t+2u)}{1+2} - \frac{\cos(u+2t-2u)}{1-2}\right]_0^t$$

$$\frac{1}{2}\left[-\frac{\cos t}{3} + \cos t + \frac{\cos(-2t)}{3}\right.$$

$$= \frac{1}{2}\int_0^t [\sin(3u-2t) + \sin(2t-u)]\,du$$

$$= \frac{1}{2}\left[-\frac{\cos(3u-2t)}{3} + \cos(2t-u)\right]_0^t$$

$$= \frac{1}{2}\left[-\frac{\cos t}{3} + \cos t + \frac{\cos(-2t)}{3} - \cos 2t\right]$$

$$= \frac{1}{2}\left[\cos t\left(\frac{1}{3}+1\right) + \cos 2t\left(\frac{1}{3}-1\right)\right.$$

$$= \frac{1}{2}\left[\frac{2\cos t}{3} - \frac{2\cos(2t)}{3}\right]$$

$$= \frac{1}{3}\left[\cos t - \cos 2t\right] \text{ Ans}$$

Que-2 Use convolution theorem to evaluate

$$L^{-1}\frac{1}{(s^2+a^2)^2}$$

let $F(s) = \frac{1}{s^2+a^2}$

$G(s) = \frac{1}{s^2+a^2}$

$$L^{-1}[F(s)] = \frac{1}{a}\sin at = f(t)$$

$$L^{-1}[G(s)] = \frac{1}{a}\sin at = g(t)$$

$$L^{-1}[F(s)\cdot G(s)] = \int_0^t f(u)\,g(t-u)\,du$$

x

$$= \int_0^t \left[\frac{1}{a}\sin au \cdot \frac{1}{a}\sin(\cancel{at-au})\right] du$$

$$= \frac{1}{2a^2}\int_0^t [2\sin au \cdot \sin(\cancel{at-au})]\, du$$

$$= \frac{1}{2a^2}\int_0^t \left[\cos(\overset{au-at+au}{\cancel{au+at+au}}) - \cos(\overset{au+at-au}{\cancel{au+at-au}})\right] du$$

$$\cancel{= \frac{1}{2a^2}\int_0^t [\cos at - \cos(2au-at)]\, du}$$

$$\frac{1}{2a^2}\int_0^t [\cos(2au-at) - \cos at]\, du.$$

$$\frac{1}{2a^2}\left[\frac{\sin(2au-at)}{2a} - u\cos at\right]_0^t$$

$$\frac{1}{2a^2}\left[\frac{\sin at}{2a} - t\cos at + \frac{\sin(at)}{2a} + 0\right]$$

$$\frac{1}{2a^2}\left[\frac{2\sin at}{2a} - t\cos at\right.$$

$$L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = \frac{1}{2a^2}\left[\frac{\sin at}{a} - t\cos at\right]$$

Que- $L^{-1}\left[\frac{s}{(s^2+4)^2}\right]$

$$F(s) = \frac{1}{s^2+2^2}$$

$$G(s) = \frac{s}{s^2+2^2}$$

$$L^{-1}[F(s)] = \frac{1}{2}\sin 2t = f(t)$$

$$L^{-1}[G(s)] = \cos 2t = g(t)$$

$$L^{-1}[F(s)\cdot G(s)] = \int_0^t f(u)\, g(t-u)\, du.$$

$$= \int_0^t \left[\frac{1}{2}\sin 2\cancel{t}u \cdot \cos(2t-2u)\right]du$$

$$= \frac{1}{4}\int_0^t [2\sin 2\cancel{t}u \cdot \cos(2t-2u)]\,du$$

$$= \frac{1}{4}\int_0^t [\sin(\cancel{2u}-\cancel{2t}+2u) + \sin(\cancel{2u}+2t-\cancel{2u})]\,du$$

$$= \frac{1}{4}\int_0^t \cancel{[\sin(2u) + \sin(2t-2u)]du}]$$

$$\frac{1}{4}\int_0^t [\sin(4u-2t) + \sin(2t)]\,du.$$

$$= \frac{1}{4}\left[-\frac{\cos(4u-2t)}{4} + u\sin 2t\right]_0^t$$

$$= \frac{1}{4}\left[-\frac{\cos 2t}{4} + t\sin 2t + \frac{\cos(-2t)}{4} + 0\right]$$

$$= \frac{1}{4}\left[\cancel{-\frac{\cos 2t}{4}} + t\sin 2t + \cancel{\frac{\cos 2t}{4}}\right]$$

$$= \frac{1}{4}t\sin 2t \quad \text{Ans}$$

Que- use convolution theorem $\frac{s^2}{(s^2+4)^2}$ to find laplace inverse

$\Rightarrow$ $L^{-1}\left[\frac{s^2}{(s^2+4)^2}\right]$

let

$F(s) = \frac{s}{s^2+2^2}$.

$G(s) = \frac{s}{s^2+2^2}$

$L^{-1}[F(s)] = \cos 2t = f(t)$

$L^{-1}[G(s)] = \cos 2t = g(t)$

$L^{-1}[F(s) \cdot G(s)] = \int_0^t f(u)\, g(t-u)\, du$

$= \frac{1}{2}\int_0^t 2\cos 2u \cos(2t-2u)\, du$

$= \frac{1}{2}\int_0^t [\cos(2u+2t-2u) + \cos(2u-2t+2u)]\, du$

$= \frac{1}{2}\int_0^t \cos(4u+2t) + \cos(-2t)\, du$

$= \frac{1}{2}\int_0^t \cos(4u+2t) + \cos(2t)\, du$.

$= \frac{1}{2}\left[\frac{\sin(4u+2t)}{4} + u\cos 2t\right]_0^t$

$= \frac{1}{2}\left[\frac{\sin 6t}{4} + t\cos 2t - \frac{\sin 2t}{4}\right]$

$= \frac{1}{2}\left[\frac{\sin 4t}{4} + t\cos 2t\right]$

Que – Use convolution theorem. s^2

(iii) $\log \frac{s^2+1}{s(s+1)}$

$$L^{-1}\left[-\frac{d}{ds}\left\{\log s^2+1 - \log s - \log s+1\right\}\right] = -t f(t)$$

$$L^{-1}\left[\frac{2s}{s^2+1} - \frac{1}{s} - \frac{1}{s+1}\right] = -t f(t)$$

$$2\cos t - 1 - e^{-t} = -t f(t)$$

$$f(t) = -\frac{2\cos t - 1 - e^{-t}}{t}$$

$$f(t) = \frac{1 + e^{-t} - 2\cos t}{t}$$ Ans

(iv) $\log\left[1 - \frac{a^2}{s^2}\right]$

$$\log\left[\frac{s^2-a^2}{s^2}\right].$$

$$L^{-1}\left[\frac{d}{ds}\left\{\log(s^2-a^2) - \log s^2\right\}\right] = -t f(t)$$

$$L^{-1}\left[\frac{1\times 2s}{s^2-a^2} - \frac{2s}{s^2}\right] = -t f(t)$$

$$L^{-1}\left[\frac{2s}{s^2-a^2} - \frac{2}{s}\right] = -t f(t)$$

$$2\cosh at - 2 = -t f(t)$$

$$f(t) = \frac{2 - 2\cosh at}{t}$$ Ans.

Que-6 $L^{-1}\left\{\frac{1}{s^3(s^2+1)}\right\} = \frac{t^2}{2} + \cos t - 1$

Sol^n $F(s) = \frac{1}{s^3}$, $G(s) = \frac{1}{(s^2+1)}$

$\cancel{L^{-1}\left[\frac{t^2}{2}\right.}$ $L^{-1}(F(s)] = \frac{t^2}{2}$ $L^{-1}[G(s)] = \sin t$

$$L^{-1}[F(s).G(s)] = \int_0^t f(u).g(t-u)\,du$$

$$= \int_0^t \frac{u^2}{2}.\ \sin(t-u)\,du.$$

$$= \frac{1}{2}\int_0^t u^2.\ \sin(t-u)\,du$$

$$= \frac{1}{2}\left[\cancel{u^2\left\{\frac{\cos(t-u)}{+1}\right\} - \frac{2u\sin(t-u)}{-1} + 2\left\{\frac{t\cos(t-u)}{+1}\right\}}\right]_0^t$$

$$= \frac{1}{2}\left[\cancel{u^2\cos(t-u) + \left(\frac{2u\sin(t-u)}{-1}\right) + 2\cos(t-u)}\right]_0^t$$

$$= \frac{1}{2}\left[\cancel{t^2 + 2 + 2\cos(-u)}\right]$$

$$= \frac{1}{2}\left[u^2\left(-\frac{\cos(t-u)}{-1}\right) - 2u\left(-\frac{\sin(t-u)}{1}\right) + 2\left(\frac{\cos(t-u)}{-1}\right)\right]_0^t$$

$$= \frac{1}{2}\left[u^2\cos(t-u) + 2u\sin(t-u) - 2\cos(t-u)\right]_0^t$$

$$= \frac{1}{2}\left[t^2 - 0 + 2t\times 0 - 0 - 2 + 2\cos t\right]$$

$$= \frac{t^2}{2} - 1 + \cos t \quad \text{Ans}$$

Que-7 State convolution theorem of laplace transform and using it find $L^{-1}\left\{\frac{1}{(s^2+4)(s+2)}\right\}$

$F(s) = \frac{1}{s^2+2^2}$ $\quad G(s) = \frac{1}{s+2}$

$L^{-1}[F(s)] = \frac{1}{2}\sin 2t$ $\quad L^{-1}[G(s)] = e^{-2t}$

$$L^{-1}[F(s).G(s)] = \int_0^t f(u)\cdot g(t-u)\,du$$

$$= \int_0^t \frac{1}{2}\sin 2u \cdot \cancel{e^{-2t}(2t-2u)}\,du$$
$$e^{(2t-2u)}\,du$$

$$= \int_0^t \frac{1}{2}\sin 2u \cdot e^{(2u-2t)}\,du$$

$$= \frac{1}{2}\int_0^t \sin 2u \cdot e^{(2u-2t)}\,du$$

$$= \frac{1}{2}\int_0^t e^{2u}\cdot e^{-2t}\sin 2u\,du.$$

$$\left\{ e^{ax}\sin bx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + C \right\}$$

$$= \frac{e^{-2t}}{2}\int_0^t e^{2u}\sin 2u\,du$$

$$= \frac{e^{-2t}}{2}\left[\frac{e^{2u}}{4+4}(2\sin 2u - 2\cos 2u)\right]_0^t$$

$$= e^{-2t}\left[\frac{e^{2u}}{8}(a\sin 2t - \cos 2t) - \frac{1}{8}(-1)\right]$$

$$= \frac{e^{-2t}}{8}\left[e^{2u}(\sin 2t - \cos 2t) + 1\right]$$

Ans

Que-04 Use convolution theorem to evaluate

$$L^{-1}\left[\frac{16}{(s-2)(s+2)^2}\right]$$

let $F(s) = \frac{16}{s-2} \rightarrow L^{-1}[F(s)] = 16e^{2t} - f(t)$

$G(s) = \frac{1}{(s+2)^2} \rightarrow L^{-1}[G(s)] = e^{-2t} L^{-1}\left(\frac{1}{s^2}\right) = te^{-2t} - g(t)$

$$L^{-1}[F(s)\cdot G(s)] = \int_0^t f(u)\cdot g(t-u)\,du.$$

$$= 16\int_0^t e^{2u}\cdot(t-u)e^{-2(t-u)}\,du.$$

$$= 16\int_0^t e^{2u-2t+2u}(t-u)\,du$$

$$= 16e^{-2t}\int_0^t \underset{I}{(t-u)}\,\underset{II}{e^{4u}}\,du$$

$$= 16e^{-2t}\left[(t-u)\frac{e^{4u}}{4} - (-1)\frac{e^{4u}}{16}\right]_0^t$$

$$= 4e^{-2t}\left[(t-u)e^{4u} + \frac{e^{4u}}{4}\right]_0^t$$

$$= 4e^{-2t}\left[0\times e^{4t} + \frac{e^{4t}}{4} - t - \frac{1}{4}\right]$$

$$= e^{-2t}\left[e^{4t} - 4t - 1\right]$$ Ans

Que - 03 Use convolution theorem to evaluate

$$L^{-1}\left[\frac{s}{(s^2+a^2)^3}\right]$$

$$F(s) = \frac{1}{(s^2+a^2)^2} \rightarrow L^{-1}[F(s)] = \cancel{e^{at} L^{-1}\left[\frac{1}{s^2}\right] e^{-at}} = f(t)$$

$=$

$$G(s) = \frac{s}{(s^2+a^2)}$$

$$L^{-1}[F(s)] = \frac{1}{2a^2}\left[\frac{\sin at}{a} - t\cos at\right] = f(t)$$

$$L^{-1}[G(s)] = \cos at$$

$$L^{-1}[F(s).G(s)] = \int_0^t \frac{1}{2a^2}\left[\frac{\sin au}{a} - u\cos au\right]\cos(at-au)\,du.$$

$$= \frac{1}{2a^2}\left[\underbrace{\frac{1}{a}\int_0^t \sin au\cos(at-au)\,du}_{I_1} - \underbrace{\int_0^t u\cos au\cos(at-au)\,du}_{I_1}\right]$$

$$I_1 = \int_0^t \sin au\cos(at-au)\,du$$

$\frac{1}{2}$

Application of Laplace Transform

2
6
7
9

⇒ Laplace Transform of Derivative :-

$$L[y'] = sL[y] - y(0)$$

$$L[y''] = s^2 L[y] - sy(0) - y'(0)$$

$$L[y'''] = s^3 L[y] - s^2 y(0) - sy'(0) - y''(0)$$

Que- Using laplace transform find the solution of the value problem.

$$\frac{d^2y}{dt^2} - 9y = 6\cos 3t \; ; \; y(0) = 2, \; y'(0) = 0$$

Sol^n $\cancel{y'' - 9y}$

$$L[y''] + 9[y] = 6L[\cos 3t]$$

$$s^2 L[y] - sy(0) - y'(0) + 9[y] = 6\frac{s}{s^2+3^2}$$

$$s^2 L[y] - 2s + 9[y] = 6\frac{s}{s^2+3^2}$$

$$(s^2+9) L[y] = 6\frac{s}{s^2+3^2} + 2s$$

$$L[y] = = 6\frac{s}{(s^2+3^2)(s^2+3^2)} + 2\frac{s}{(s^2+3^2)} \quad \text{———} \; (1)$$

$$y = 6L^{-1}\left[\frac{s}{(s^2+3^2)(s^2+3^2)}\right] + 2L^{-1}\left[\frac{s}{s^2+3^2}\right] \quad \text{———} \; (1)$$

$$L^{-1}\left[\frac{s}{(s^2+3^2)(s^2+3^2)}\right]$$

$$= \text{let } F(s) = \frac{1}{(s^2+3^2)} \rightarrow L^{-1}[F(s)] = \frac{1}{3}\sin 3t$$

$$G(s) = \frac{s}{(s^2+3^2)} \Rightarrow L^{-1}[G(s)] = \cos 3t.$$

$$L^{-1}[F(S).\ G(S)] = \int_0^t f(u).g(t-u)\,du$$

$$= \frac{1}{3}\int_0^t \sin 3u \cos(3t-3u)\,du$$

$$= \frac{1}{6}\int_0^t 2\sin 3u \cos(3t-3u)\,du$$

$$= \frac{1}{6}\int_0^t [\sin(3u-3t+3u) + \sin(3u+3t-3u)]\,du.$$

$$= \frac{1}{6}\int_0^t [\sin(6u-3t) + \sin 3t]\,du$$

$$= \frac{1}{6}\left[-\frac{\cos(6u-3t)}{6} + u\sin 3t\right]_0^t$$

$$= \frac{1}{6}\left[-\frac{\cos 3t}{6} + t\sin 3t + \frac{\cos(-3t)}{6} + 0\right]$$

$$6L^{-1}\left[\frac{s}{(s^2+3^2)(s^2+3^2)}\right] = \frac{1}{6}[t\sin 3t]$$

$$2L^{-1}\left[\frac{s}{s^2+3^2}\right] = \cos 3t$$

$$y = \underline{t\sin 3t + 2\cos 3t}$$

Que-02 Using convalution solve the initial value problem $\frac{d^2y}{dt^2} + 9y = \sin 3t$, given $y(0) = 0, \frac{dy}{dt} = 0$

$$\boxed{L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = \frac{1}{2a^2}\left[\frac{\sin at}{a} - t\cos at\right]}$$

$$y'' + 9y = \sin 3t$$

$$s^2L[y] - sy(0) - y'(0) + 9L[y] = L[\sin 3t]$$

$$L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = \frac{1}{2a^2}\left[\frac{\sin at}{a} - t\cos at\right]$$

$$(s^2+9)L[y] = \frac{3}{s^2+3^2}$$

$$L[y] = \frac{3}{(s^2+3^2)(s^2+3^2)}$$

$$y = 3L^{-1}\left[\frac{1}{(s^2+3^2)(s^2+3^2)}\right]$$

$L[t^2]$

$$3\left[\frac{1}{2\times 3^2}\left\{\frac{\sin 3t}{3} - t\cos 3t\right\}\right.$$

$$Y = \Rightarrow \frac{1}{6}\left[\frac{\sin 3t}{3} - t\cos 3t\right] \text{ Ans}$$

Que - 7/ Using L.T to solve the diff. eqn $\frac{d^2y}{dt^2} + 9y = \cos 2t$

where $y(0) = 1$ $y(\frac{\pi}{2}) = -1$ $y'(0) = K$

or

$$y'' + 9y = \cos 2t \qquad \frac{d^2x}{dt^2} + 9x = \cos 2t = x(0) = 1$$

$$x(\tfrac{\pi}{2}) = -1$$

$$x'(0) = K$$

$$s^2 L[y] - s\,y(0) - y'(0) + 9L[y] = L[\cos 2t]$$

Sol $(s^2+9)L[y] - s - K = \frac{s}{s^2+2^2}$

$$(s^2+9)L[y] = \frac{s}{s^2+2^2} + s + K$$

$$L[y] = \frac{s}{(s^2+2^2)(s^2+3^2)} + \frac{s}{s^2+3^2} + K\frac{1}{s^2+3^2}$$

$$y = L^{-1}\left[\frac{s}{(s^2+2^2)(s^2+3^2)} + \cos 3t + \frac{K}{3}\sin 3t\right] \longrightarrow (1)$$

$$= L^{-1}\left[\frac{s}{(s^2+2^2)(s^2+3^2)}\right]$$

let $F(s) = \frac{1}{(s^2+2^2)} \rightarrow L^{-1}[F(s)] = \frac{1}{2}\sin 2t \rightarrow f(t)$

$$G(s) = \frac{s}{s^2+3^2} \rightarrow L^{-1}[G(s)] = \cos 3t \rightarrow g(t)$$

$$L^{-1}[F(s).G(s)] = \int_0^t f(u)\,g(t-u)\,du$$

$$= \frac{1}{2}\int_0^t \sin 2u . \cos(3t - 3u)$$

$$= \frac{1}{4}\int_0^t 2\sin 2u \cdot \cos(3t-3u)\,du$$

$$= \frac{1}{4}\int_0^t \left[\sin(2u-3t+3u) + \sin(2u+3t-3u)\right]du$$

$$= \frac{1}{4}\int_0^t \left[\sin(5u-3t) + \sin(3t-u)\right]du.$$

$$= \frac{1}{4}\left[-\frac{\cos(5u-3t)}{5} + \left\{\frac{+\cos(3t-u)}{+1}\right\}\right]_0^t$$

$$= \frac{1}{4}\left[-\frac{\cos 2t}{5} + \cancel{\cos 2t} + \frac{\cos(-3t)}{5} - \cancel{\cos 3t}\right]$$

$$= \frac{1}{4}\left[-\frac{\cos 2t}{5} + \frac{\cos 3t}{5} + \cos 2t - \cos 3t\right]$$

$$= \frac{1}{4}\left[\frac{4\cos 2t}{5} - \frac{4}{5}\cos 3t\right]$$

$$= \frac{1}{5}\left[\cos 2t - \cos 3t\right]$$

$$y = \frac{1}{5}\left[\cos 2t - \cos 3t\right] + \cos 3t + \frac{K}{3}\sin 3t$$

$$y = \frac{1}{5}\cos 2t - \frac{1}{5}\cos 3t + \cos 3t + \frac{K}{3}\sin 3t$$

$$y = \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t + \frac{K}{3}\sin 3t \quad \text{——— (2)}$$

$$y = -1 \quad t = \frac{\pi}{2}$$

$$-1 = \frac{1}{5}\cos 2\frac{\pi}{2} + \frac{4}{5}\cos 3\frac{\pi}{2} + \frac{K}{3}\sin 3\frac{\pi}{2}$$

$$-1 = -\frac{1}{5} + \frac{K}{3}$$

$$\frac{K}{3} = -\frac{1}{5} + 1$$

$$\frac{K}{3} = \frac{4}{5} \Rightarrow K = \frac{12}{5}$$

From eqn (2)

$$y = \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t + \frac{4}{5}\sin 3t$$

Ans

Que-01 Solve using laplace transform $y''+4y'+4y=6e^{-t}$

$y(0)=-2, \quad y'(0)=8$

$$s^2L[Y]-sY(0)-y'(0)+\cancel{4L[s}\ 4[s\,L[Y]-Y(0)]+4L[Y]=L[6e^{-t}]$$

$$s^2L[Y]-sY(0)-y'(0)+4\{sL[Y]-Y(0)\}+4L[Y]=L[6e^{-t}]$$

$$=(s^2+4s+4)L[Y]+2s-\cancel{8}+\cancel{8}=\frac{6}{s+1}$$

$$=(s^2+4s+4)L[Y]=6\frac{1}{s+1}-2s+\cancel{0}$$

$$(s+2)^2L[Y]=6\frac{1}{s+1}-2s$$

$$L[Y]=6\frac{1}{(s+1)(s+2)^2}-2\frac{s}{(s+2)^2}$$

$$Y=6L^{-1}\left[\frac{1}{(s+1)(s+2)^2}\right]-2L^{-1}\left[\frac{s}{(s+2)^2}\right]$$

$$L^{-1}\left[\frac{1}{(s+1)(s+2)^2}\right]$$

let $F(s)=\frac{1}{s+1}\rightarrow e^{-t}L^{-1}\left[\frac{1}{s}\right]\Rightarrow e^{-t}\rightarrow f(t)$

$$G(s)=\frac{1}{(s+2)^2}\rightarrow e^{-2t}L^{-1}\left[\frac{1}{s^2}\right]=te^{-2t}\rightarrow g(t)$$

$$L^{-1}[F(s)\cdot G(s)]=\int_0^t f(u)\cdot g(t-u)\,du$$

$$=\int_0^t e^{-u}\cdot(t-u)\,e^{-2(t-u)}\,du$$

Que- solve the following differential eqn. using Laplace transform

$$\frac{d^3y}{dt^3} - \frac{3d^2y}{dt^2} + 3\frac{dy}{dt} - y = t^2e^t \rightarrow \quad y(0) = 1$$

$$y''' - 3y'' + 3y' - y = e^t t^2 \qquad y'(0) = 0$$

$$y''(0) = -2$$

$$s^3L[y] - s^2y(0) - sy'(0) - y''(0) - 3s^2L[y] + 3sy(0) + 3y(0)$$

$$+ 3sL[y] - 3y(0)$$

$$-L[y] = \frac{2}{(s-1)^3}$$

$$L[y](s^3 - 3s^2 + 3s - 1) - s^2 + 2 + 3s - 3 = \frac{2}{(s-1)^3}$$

$$L[y](s-1)^3 = \frac{2}{(s-1)^3} + s^2 - 3s + 1$$

$$L[y] = \frac{2}{(s-1)^6} + \frac{s^2 - 3s + 1}{(s-1)^3}$$

$$y = 2L^{-1}\left[\frac{1}{(s-1)^6}\right] + L^{-1}\left[\frac{s^2 - 2s + 1 - s}{(s-1)^3}\right] \Rightarrow L^{-1}\left[\frac{(s-1)^2}{(s-1)^3} - \frac{s}{(s-1)^3}\right]$$

$$= 2L^{-1}\left[\frac{1}{(s-1)^6}\right] + L^{-1}\left[\frac{1}{s-1} - \frac{1}{(s-1)^2} - \frac{1}{(s-1)^3}\right]$$

$$= \frac{2e^t t^5}{120} + e^t - te^{2t} - t^2e^{3t}$$

Ans

$$y = 2L^{-1}\left[\frac{1}{(s-1)^6}\right] + L^{-1}\left[\frac{s^2 - 2s + 1 - s}{(s-1)^3}\right]$$

$$= 2e^t\frac{t^5}{5!} + L^{-1}\left[\frac{1}{s-1}\right] - L^{-1}\frac{(s-1)+1}{(s-1)^3}$$

$$y = e^t\frac{t^5}{60} + e^t - L^{-1}\left[\frac{1}{(s+1)^2}\right] - L^{-1}\left[\frac{1}{(s-1)^3}\right]$$

$$y = \frac{t^5e^t}{60} + e^t - te^t - e^t\frac{t^2}{2}$$ Ans.

Que-03 Using laplace transform solve the diff. eqn $\frac{d^2x}{dt^2}+2$

$$\frac{d^2x}{dt^2}+2\frac{dx}{dt}+5x=e^{-t}\sin t,\quad x(0)=0,\quad x'(0)=1$$

$$x''+2x'+5x=L[e^{-t}\sin t]$$

$$s^2L[x]-sx(0)-x'(0)+2sL[x]-2x(0)+5L[x]=L[e^{-t}\sin t]$$

$$L[x](s^2+2s+5)-1=\frac{1}{(s+1)^2+1}$$

$$L[x]=\left[\frac{1}{(s^2+2s+2)(s^2+2s+5)}+\frac{1}{(s^2+2s+5)}\right]$$

$$[x]=L^{-1}\left[\frac{1}{(s^2+2s+2)(s^2+2s+5)}\right]+L^{-1}\left[\frac{1}{s^2+2s+5}\right]$$

$$=L^{-1}\left[\frac{1}{[(s+1)^2+1][(s+1)^2+2^2]}\right]+L^{-1}\left[\frac{1}{(s+1)^2+2^2}\right]$$

$$=e^{-t}L^{-1}\left[\frac{1}{(s^2+1)(s^2+2^2)}\right]+e^{-t}\frac{1}{2}\sin 2t.$$

let $F(s)=\frac{1}{(s^2+1)}\rightarrow L^{-1}[F(s)]=\sin t\rightarrow f(t)$

$G(s)=\frac{1}{(s^2+2^2)}\rightarrow L^{-1}[G(s)]=\frac{1}{2}\sin 2t$

$$L^{-1}[F(s).G(s)]=\int_0^t f(u)\,g(t-u)\,du.$$

$$=\frac{1}{2}\int_0^t \sin u\cdot\sin(2t-2u)\,du.$$

$$=\frac{1}{4}\int_0^t 2\sin u\cdot\sin(2t-2u)\,du$$

$$=\frac{1}{4}\int_0^t[\cos(u-2t+2u)-\cos(u+2t-2u)]\,du.$$

$$=\frac{1}{4}\int_0^t[\cos(3u-2t)-\cos(2t-u)]\,du.$$

$$=\frac{1}{4}\left[\frac{\sin(3u-2t)}{3}+\frac{\sin(2t-u)}{-1}\right]_0^t$$

$$= \frac{1}{4}\left[\cancel{\sin 2t}\ \frac{\sin t}{3} + \sin t + \frac{\sin 2t}{3} - \sin 2t \right] \quad \frac{1}{3} - 1$$

$$= \frac{1}{4}\left[\frac{4}{3}\sin t - \frac{2}{3}\sin 2t \right]$$

$$= \frac{1}{6}\left[2\sin t - \sin 2t \right]$$

$$x = \frac{e^{-t}}{6}\left[2\sin t - \sin 2t\right] + e^{-t}\frac{1}{2}\sin 2t \quad \text{Ans.}$$